A Fast Direct Solver for Higher Order Discretizations of Integral Equations

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***Abstract*—This paper presents a fast direct solver for the Combined Field Integral Equation using higher-order discretizations. By adopting higher-order polynomials with the Method of Moments, the number of unknowns is significantly reduced. The fast direct solver leverages the efficiency of the Multi Level Fast Multipole Method by combining it with randomized linear algebra to construct low-rank approximations in a $\mathcal{H}^2$ format. The proposed method is fully error controllable and achieves a setup time with computational complexity of $\mathcal{O}(N + r^3 \log N)$. Numerical results for the scattering problem of a sphere demonstrate high accuracy, and the efficiency is demonstrated on the NASA Almond.**

I. Introduction

Solving the Electric Field Integral Equations (EFIE) or Magnetic Field Integral Equations (MFIE) involves discretizing surface density currents and solving a large dense linear system of N unknowns when using the widely employed Method of Moments (MoM). Selecting an efficient discretization is essential for performance. The most popular discretization is the RWG [1] first-order basis functions. However, [2] showed that it is possible to drastically reduce the number of unknowns N by utilizing higher-order (HO) polynomials to represent the surface currents and the geometry.

The reduced number of unknowns aids in solving high frequency problems, however, conventional full-wave solvers suffer from high complexities, ranging from $\mathcal{O}(N^2)$ to $\mathcal{O}(N^3)$. Therefore, accelerated methods such as the Multi-Level Fast Multipole Method (MLFMM) [3] have been proposed.

More recently, [4] introduced the algebraic counterpart of MLFMM, Hierarchical matrices, specifically the $\mathcal{H}^2$ format. The hierarchical matrix format relies on algebraic compression of matrix blocks of the MoM matrix and opens the possibility of solving scattering problems with fast direct solvers, completely avoiding iterative solvers. However, computing an efficient representation of high-frequency integral equations is still an active research area.

Some approaches use skeletonization to describe the interaction of distant groups, only using a subset of the rows and columns. These methods are, however, not well suited for higher-order basis functions, as matrix elements are usually generated in blocks, resulting in an inefficient compression. Moreover, error control is generally more difficult for skeletonization methods and they often provide worse approximations for a given rank r than sketching methods [5].

Our work concerns building a fast direct solver using the $\mathcal{H}^2$ format and utilizing higher-order basis functions. To achieve this, we combine randomized linear algebra and MLFMM specialized for HO basis functions [6]. Hereby we achieve a setup time with computational complexity of $\mathcal{O}((r^3 + Nr) \log N)$. In practice, the rank r usually scales proportional to $N^{1/2}$ for electrically large quasi-planar geometries [7].

II. Theory

A $\mathcal{H}^2$ matrix is used to efficiently represent the MoM matrix $\boldsymbol{Z}$. This matrix format exploits the block low-rank structure resulting from the singular Helmholtz kernel. To express the block low-rank structure, a cluster tree is used to recursively partition the domain of unknowns into clusters of constant size. A $\mathcal{H}^2$ matrix compresses entire block rows and columns, such that the block between clusters s and t is represented as:

$$\boldsymbol{Z}_{st} \simeq \boldsymbol{Q}_s \boldsymbol{B}_{st} \boldsymbol{V}_t^H. \tag{1}$$

A. Sketching with MLFMM

We compute the orthogonal bases of rows and columns in (1) by utilizing randomized linear algebra and MLFMM, which approximates matrix-vector products as

$$\boldsymbol{Z}\boldsymbol{v} \simeq \boldsymbol{Z}_{\text{near}}\boldsymbol{v} + \boldsymbol{Z}_{\text{far}}\boldsymbol{v}, \tag{2}$$

where $\boldsymbol{Z}_{\text{near}}$ is sparse and $\boldsymbol{Z}_{\text{far}}$ can be applied efficiently in $\mathcal{O}(N \log N)$ time. The orthonormal row and column bases $\boldsymbol{Q}$ and $\boldsymbol{V}$ are computed as the range of the sketch:

$$\boldsymbol{Y} = \boldsymbol{Z}_{\text{far}}\boldsymbol{\Omega}, \qquad \boldsymbol{X} = \boldsymbol{Z}_{\text{far}}^H \boldsymbol{\Psi}, \tag{3}$$

where $\boldsymbol{\Omega}, \boldsymbol{\Psi} \in \mathbb{R}^{N \times r}$ and the entries are i.i.d. normal distributed [8]. Products with $\boldsymbol{Z}_{\text{far}}$ allows for sketching of all rows/columns simultaneously, and the sketching is repeated for each level in a bottom-up fashion. Sketching of the entire matrix requires $\mathcal{O}(r \log N)$ matrix-vector products in total.

B. Factorizing the matrix

A common strategy for factorizing $\mathcal{H}^2$ matrices aims at finding a transformation such that the problem implicitly becomes a sparse factorization problem [9]. The used factorization utilizes that all coupling matrices $\boldsymbol{B}_{st}$ are orthogonal to the nullspace of the row and column bases $\boldsymbol{Q}_s, \boldsymbol{V}_t$

$$\begin{pmatrix} \boldsymbol{Q}_s & \boldsymbol{Q}_s^\perp \end{pmatrix}^H \boldsymbol{Q}_s \boldsymbol{B}_{st} \boldsymbol{V}_t^H \begin{pmatrix} \boldsymbol{V}_t & \boldsymbol{V}_t^\perp \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}_{st} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}, \tag{4}$$

thus allowing for efficient elimination of all sparsified indices. This is then repeated for each group in each level, until we are left with a small dense matrix which needs to be factorized.

III. Numerical results

All results presented in this paper are done on a computer with an Intel Xeon Gold 5218 CPU with 32 cores and 256 GB memory. All test-case geometries have been discretized by quadrilatterals of side length at most 1.5λ and the highest order of there basis functions are of 5th order. We set the desired relative error tolerance of the fast direct solver to 10^{-3} in all computations.

A. Sphere

As a test case, we consider the scattered field from a sphere with diameter 20λ illuminated by a plane wave at 6 GHz. The scattered field is computed by solving the CFIE at an accuracy of 10^{-3} and is compared to a Mie series, which is an exact reference solution. The numerical solution achieves a relative RMS (RRMS) of 0.3%, and the resulting scattered fields are shown in Figure 1, where we see that the two solutions are almost indistinguishable.

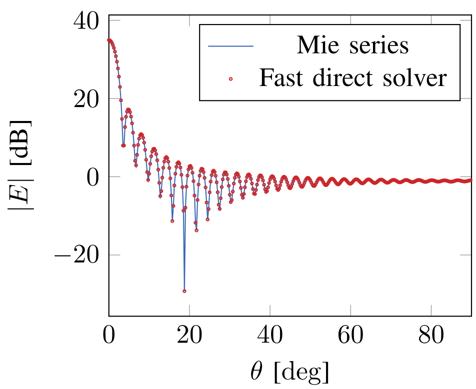


Fig. 1: Comparison of the scattered field on a 20λ sphere between the numerical solution of the fast direct solver and the Mie series solution.

B. NASA Almond

We investigate the efficiency of the algorithm by computing the scattered field of an increasingly larger NASA almond shown in Figure 2. The object is 0.25 m long and 0.1 m wide, being illuminated by a plane wave with frequency varying from 20 to 150 GHz. This leads to systems of equations with the number of HO unknowns ranging from 10k to 380k. Note that the discontinuous contributions are not fully accounted for during filling. However, tests have shown that we can achieve a RRMS of 0.08% compared to the MoM solution.

The numerical results in Figure 3a show that a scaling proportional to $N^{3/2}$ in the computational time is achieved. This is the best possible scaling that we can expect for geometry which is mostly locally quasi-planar.

The memory required for factorization storage is shown in Figure 3b. The memory scales proportionally to $N^{3/2}$, which is larger than the expected $N \log N$ and is probably due to the rank of the interactions between the two planar surfaces of the almond which scales higher than $N^{1/2}$.

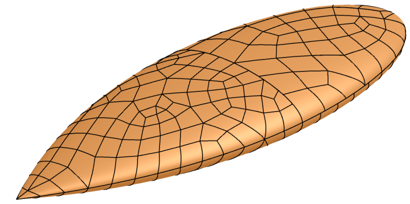
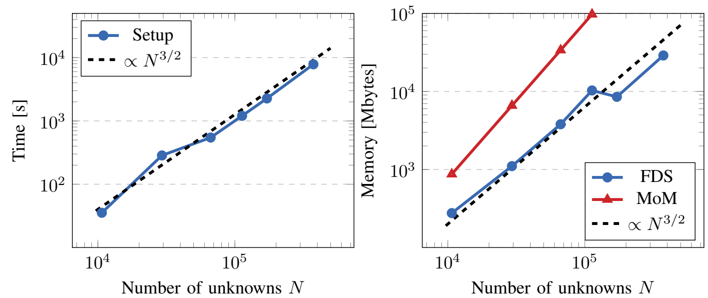


Fig. 2: The NASA almond meshed with HO quadrilaterals.



(a) (b)

Fig. 3: Scaling of the fast direct solver in time and memory for different electrical sizes of the NASA almond.

IV. Conclusion

A fast direct solver for higher-order basis functions has been implemented. This was achieved by the construction and factorization of a $\mathcal{H}^2$ matrix using randomized linear algebra. The results demonstrate its efficiency in compressing higher-order basis functions even for electrically large nontrivial problems.

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