Uncertainty Quantification for Reflectarrays

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***Abstract*—Reflectarray antennas for space applications involve a large number of elements with varying sizes to be printed on a dielectric substrate. Due to the manufacturing process, the fabricated reflectarray might have significant deviations from the desired design. This paper demonstrates an efficient method that allows the antenna designer to accurately predict the impact of manufacturing deviations on the performance of the reflectarray.**

I. Introduction

The use of reflectarrays for space applications has been studied intensively in recent years, giving rise to a range of potential application scenarios where reflectarrays and other quasi-periodic structures are more appealing than traditional antenna designs such as reflector antennas.

For space applications, quantifying the uncertainty in performance of the produced reflectarray is critical to ensure that the mission is succesful. In this paper, we make use of modern tools from the mathematical uncertainty quantification theory and combine them with state-of-the-art simulation software for quasi-periodic antennas to allow accurate and efficient uncertainty quantification.

II. Mathematical Uncertainty Quantification

Uncertainty Quanfication (UQ) has been a subject of study in the applied mathematics community for several decades, and has in the recent years seen a significant increase in interest, particularly due to progress made in areas such as stochastic collocation.

The fundamental question that UQ considers is:

> Given a system where the behaviour can be modelled as $F(\overline{X})$, where $\overline{X}$ is a vector of D stochastic elements, characterize the uncertainty of $F(\overline{X})$.

In this formulation, characterizing the uncertainty typically involves gathering data such that the following questions can be answered:

- **Mean performance:** What is the expected performance of the system?
- **Variation in performance:** What is the variation in the performance of the system?
- **Deviation from nominal:** How does the expected performance deviate from the intended performance?
- **Confidence intervals:** With α-percent certainty, how will the final system perform?

The traditional approach to answering these question is the so-called Monte-Carlo method, which simply samples the function F a large number of times by picking $\overline{X}$ according to the distribution of its elements, and then looks at the distribution of the output.

Aside from being very simple to implement, the Monte-Carlo method has the theoretical advantage that its convergence rate is independent of the number of parameters D in the $\overline{X}$ vector. The main drawback, however, of Monte-Carlo is that in practice, the obtained statistical estimates are too inaccurate, particularly when applying costly simulation methods and/or when considering sensitive output parameters such as return loss, cx-polar performance, etc.

An alternative is the approach known as stochastic collocation, which approximates the moments of the function F by numerical integration:

$$\textbf{Expected value}: \quad \mathrm{E}(F(\overline{X})) = \mu_F \tag{1}$$
$$= \int_{a^{(1)}}^{b^{(1)}} \int_{a^{(2)}}^{b^{(2)}} \dots \int_{a^{(D)}}^{b^{(D)}} g_X(\overline{X}) F(\overline{X}) d\overline{X},$$
$$\textbf{Variance}: \quad \mathrm{Var}(F(\overline{X})) = \sigma_F^2 \tag{2}$$
$$= \int_{a^{(1)}}^{b^{(1)}} \int_{a^{(2)}}^{b^{(2)}} \dots \int_{a^{(D)}}^{b^{(D)}} g_X(\overline{X}) F^2(\overline{X}) d\overline{X} - \mu_F^2.$$

In this notation, g_X is the D-dimensional distribution function for X, and $a^{(i)}$ and $b^{(i)}$ are the limits of this distribution function. The most common distributions are shown below:

Distribution	$g_X(x)$	$[a, b]$
Normal	$\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$	$[-\infty, \infty]$
Uniform	$\frac{1}{2}$	$[-1, 1]$
Exponential	e^{-x}	$[0, \infty]$

A. Numerical implementation

Computing the D-dimensional integrals (1) and (2) in an efficient manner is a key factor in achieving satisfactory performance when compared to e.g. Monte-Carlo. It is important to stress that while an efficient implementation of any UQ algorithm requires careful consideration of many numerical details, the dominant factor in performance for our application is implementation of efficient multi-dimensional integration rules. This is particularly true as D increases and as $F(\overline{X})$ has a more complex behaviour.

Efficient multi-dimensional integration grids can be obtained through the use of *sparse grids*, which employ Smolyak's l'th order tensor combination of one-dimensional

TABLE I
REFLECTARRAY DATA

Center frequency	10 GHz
Frequency range	$9 - 11$ GHz
Number of elements	50×50
Reflectarray dimensions	600 mm $\times$ 600 mm
Relative permittivity	$\epsilon_r = 3.66$
Substrate thickness	$d = 1.524$ mm
Loss tangent	$\tan\delta = 0.0037$

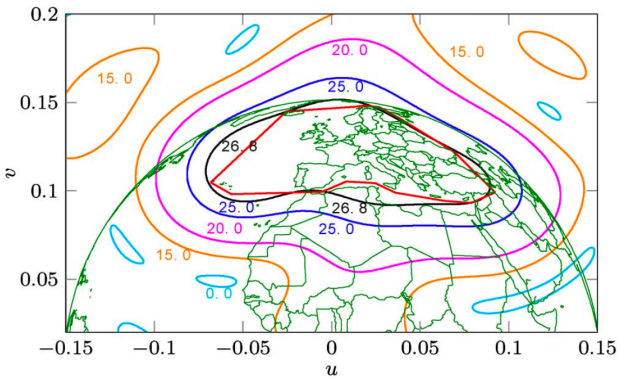

Fig. 1. Optimized co-polar radiation patterns of the reflectarray specified in Table I.

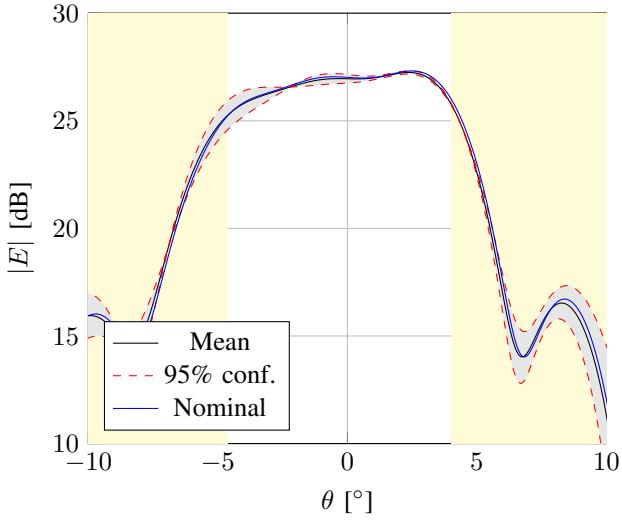

Fig. 2. Mean, 95% confidence interval and nominal ($\epsilon_d = 3.66$) pattern for the reflectarray when the relative permittivity ϵ_d of the dielectric substrate is uniformly distributed $U(3.56, 3.76)$. The deviations in the pattern are substantial, and thus the performance of the reflectarray is quite sensitive to ϵ_d. Marked in yellow is the coverage area (the red zone in Fig. 1.

rules [1, Section 3.2.2]. The computational complexity of algorithms based on sparse grids is discussed in detail in [2], but suffice it to say that the number of integration points scales as $2^l \frac{N^D}{D!}$, where N is the order of the one-dimensional sampling rule and l is the order of the grid. Since $2^l \ll D!$, sparse grids yield far fewer points that the traditional tensor product rules where the number of points scale as N^D.

For $D \gtrsim 100$, even sparse grids can require too many integration points. In particular, if $F(\overline{X})$ behaves reasonably simple in the domain of interest (for instance because the quality of the manufacturing process is high enough for the intended application such that the output is not too sensitive), the high accuracy of grid-based methods such as sparse grids might not be necessary. In this implementation, we apply the Stroud rules [3] of second and third order, which require $D+1$ and $2D$ evaluations, respectively, and consider whether the integral has converged based on these two rules. If the integral has not converged, we then apply sparse grids.

III. RESULTS

We consider a reflectarray previously discussed in the literature [4], a planar array with specifications as described in Table I The elements have been optimized to provide a high gain over a European coverage, as shown in Fig. 1.

We begin by considering the effect of adding an uncertainty to the size of the square elements, such that a small deviation from the designed sizes are added, uniformly distributed in the range ± 0.03 mm, resulting in $D = 2500$ variables. We then consider the resulting uncertainty in the co-polar and cx-polar levels, applying the UQ algorithm requiring 7501 evaluations to converge. Due to the fast simulation method as described in [4], this task requires last than two hours of total computation time on a laptop.

The resulting 95% confidence interval for the minimum directivity in the coverage is 27.3 ± 0.02 dB, in other words, a very low sensitivity of the directivity to small uncertainties in the manufacturing. This allows us to conclude that imperfections in the printing of the elements will not significantly impact the performance of the array, a conclusion that can be very important in the design phase of an antenna system.

We then move to consider uncertainty of the relative permittivity of the dielectric substrate, for ϵ_r distributed as $U(3.56, 3.76)$, which means $D = 1$ variable and thus the complete uncertainty quantification is completed in less than a minute. The resulting uncertainty is shown in Fig. 2, clearly demonstrating that the value of the relative permittivity significantly impacts the resulting pattern.

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