Uncertainty Quantification for Reflector Antennas

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Abstract—Large reflector antennas for telecommunication or space observation purposes often require a lengthy and detailed design process, where the design is carefully revised to ensure the best performance for the intended application. For many modern reflector antennas, quantification of the uncertainty in performance as a function of manufacturing errors is an important part of the design process. In this paper, we present an efficient way of quantifying the effects of uncertainty in the production. The method is shown to far outperform the conventional Monte-Carlo techniques in accuracy and computational time.

Index Terms-uncertainty quantification, reflector antennas.

I. INTRODUCTION

Designing reflector antenna systems for modern telecommunication applications entails very stringent performance requirements and strict error budgets. As the systems become increasingly complex and involve many subsystems, the need for accurate and reliable quantification of the errors involved in the error budgets becomes greater and greater.

Modern computational electromagnetics software makes it possible for the engineers behind such systems to simulate a large number of design choices, in some cases even allowing fully automatic optimization to attain optimal performance. However, when it comes to quantifying the uncertainty, i.e., the dependency of the performance on the mechanical errors in the system when it is produced, the engineers are on their own. In practice, many engineers will simply build the system, measure it, and compare the results and try to explain the discrepancies — this approach means that the engineer does not have full control of the final performance.

If the engineers apply some form of uncertainty analysis, most will resort to simply running a very large number of simulations with random errors added sporadically to the system, and then perform some statistical examination on that data. The downsides to this approach are clear: A very large number of simulations is required, and the risk of user error is high.

This paper presents an alternative approach. The user is required to specify how the errors manifest themselves in the system, e.g. surface errors in the reflectors, inaccurate mounting or phase errors in the feed, and so on. Based on this input, the algorithm automatically determines the uncertainty of the output parameter of interest.

The paper is divided as follows. After a discussion of mathematical Uncertainty Quantification (UQ) in Section II, including a discussion of how these algorithms greatly outperform traditional methods based on Monte-Carlo sampling,

we discuss how to implement a specific algorithm efficiently in Section II-A. After this discussion, we present a number of test cases in Section III that illustrate how the method is able to quantify the performance uncertainty caused by geometrical errors in the design.

II. MATHEMATICAL UNCERTAINTY QUANTIFICATION

Uncertainty Quanfication (UQ) has been a subject of study in the applied mathematics community for several decades, and has in the recent years seen a significant increase in interest, particularly due to progress made in areas such as stochastic collocation.

The fundamental question that UQ attempts to answer is:

Given a function $F(\overline{X})$, where \overline{X} is a vector of D stochastic elements, characterize the behaviour of $F(\overline{X})$.

How to characterize the behavour, however, is not straightforward. The traditional approach is the so-called Monte-Carlo method, which simply samples the function F a large number of times by picking \overline{X} according to the distribution of its elements. Aside from being very simple to implement, the Monte-Carlo method has the advantage that its convergence rate is $\frac{1}{\sqrt{M}}$, where M is the number of evaluations of F, and thus the convergence speed is independent of the number of parameters D in the \overline{X} vector.

The drawback, however, of Monte-Carlo is that in practice, $\frac{1}{\sqrt{M}}$ can be too slow for many applications, in particular scenarios where high accuracy is needed and/or where the number of parameters D is not too large. In these cases, higher-order methods such as stochastic collocation offer a far better convergence rate for a moderate number of parameters D, while only being slightly more complicated to implement.

Stochastic collocation is done by approximating the moments of the function F by numerical integration:

Expected value:
$$E(F(X)) = \mu_F$$
 (1)

$$= \int_{a^{(1)}}^{b^{(1)}} \int_{a^{(2)}}^{b^{(2)}} \dots \int_{a^{(D)}}^{b^{(D)}} g_X(\overline{X}) F(\overline{X}) d\overline{X},$$
Variance: $Var(F(\overline{X})) = \sigma_F^2$ (2)

$$= \int_{a^{(1)}}^{b^{(1)}} \int_{a^{(2)}}^{b^{(2)}} \dots \int_{a^{(D)}}^{b^{(D)}} g_X(\overline{X}) F^2(\overline{X}) d\overline{X} - \mu_F^2.$$



Fig. 1. The approximation to the true F(X) achieved by applying Monte-Carlo and our higher-order UQ collocation, with each method evaluating F(X) at 7 points. Clearly, the higher-order method achieves a much better approximation and subsequently a more accurate integral when applied to (1) and (2).

In this notation, $g_{X^{(i)}}$ is the distribution function for the *i*'th element in X, and $a^{(i)}$ and $b^{(i)}$ are the limits of this distribution function. The most common distributions are shown below:

Distribution	$g_X(x)$	[a,b]
Normal	$\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$	$[-\infty,\infty]$
Uniform	$\frac{1}{2}$	[-1, 1]
Exponential	\tilde{e}^{-x}	$[0,\infty]$

With these moments of the function, a range of statistical estimates can be obtained:

- Mean performance: What is the expected (mean) performance of the system?
- Variation in performance: What is the variation in the performance of the system?
- **Deviation from nominal:** How does the expected performance deviate from the nominal performance?
- **Confidence intervals:** With *α*-percent certainty, how will the system perform?

These estimates are often required when designing antennas for industrial applications.

The difference in practice between the convergence rate of the two classes of methods, Monte-Carlo (MC) and higherorder methods is extreme for a small number of unknowns. An illustration of the difference is shown in Figure 1, where the true function F is shown in black, and in red and green is shown the approximation by the MC and higher-order method, respectively, based on 7 evaluations of F. Clearly, the accuracy of the higher-order method is much better than MC.

A. Numerical implementation

As shown in Figure 1, the improvement in performance is caused by performing the integration using higher-order integration methods, approximating the integrand by a polynomial basis. In this paper we will apply collocation methods, which simply sample the function F(X) in the way that is most appropriate for the distribution function for the elements in X, and perform the integrals (1)-(2) by numerical quadrature. We will give a brief, non-mathematical sketch of the approach below, but refer readers to [1], [2].

Computing the *D*-dimensional integrals (1) and (2) in an efficient manner is a key factor in achieving satisfactory performance. It is important to stress that while an efficient implementation of any UQ algorithm requires careful consideration of many numerical details, the dominant factor in performance for our application is implementation of efficient multi-dimensional integration rules. This is particularly true as *D* increases and as $F(\overline{X})$ begins to have a more complex behaviour.

For moderate values of D, say, $D \leq 100$, naive UQ implementations will apply tensor grids, constituting a tensor product of a set of one-dimensional rules. Tensor grids require an unacceptable amount of function evaluations for large D, since the number of integration points scale as N^D , where N is the number of integration points for the one-dimensional rule.

Efficient multi-dimensional integration grids can instead be obtained through the use of *sparse grids*, which employ Smolyak's *l*'th order tensor combination of one-dimensional rules [1, Section 3.2.2]. The computational complexity of algorithms based on sparse grids is discussed in detail in [3], but suffice it to say that the number of integration points scales as $2^{l} \frac{N^{D}}{D!}$. Since $2^{l} \ll D!$ for all but trivial problem sizes, sparse grids provide a significant reduction in the number of function evaluations compared to tensor grids.

For $D \gtrsim 100$, even sparse grids can begin to require too many evaluations. In particular, if $F(\overline{X})$ behaves reasonably simple in the domain of interest (for instance because the quality of production is high enough for the intended application), the high accuracy of grid-based methods might not be necessary. Instead, simpler rules can be applied as a first attempt. In this implementation, we apply the Stroud rules [4] of second and third kind, which require D + 1 and 2Devaluations, respectively, and consider whether the integral has converged based on these two evaluations. If the integral has not converged, we then apply sparse grids.

III. RESULTS

In the following, we will consider several examples to illustrate the applicability of uncertainty quantification. All simulations have been performed on a laptop with an 2.6 GHz i7 processor and 16 GB RAM.

A. Comparison: Radiation from a circular aperture

We begin by looking at the radiation from a circular aperture mounted on an infinite ground plane, letting the radius of the aperture be a stochastic variable with uniform distribution.



Fig. 2. Comparison between the number of function evaluations required for a specific accuracy in the estimate of the mean by the UQ algorithm and a traditional Monte-Carlo implementation. The case is discussed in Section III-A.

Based on [5, (6-131)], the θ -polarized electric far field from a circular aperture of radius X is

$$E(X; r, \theta, \phi) = j \frac{kX^2 E_0 e^{-jkr}}{r} \left(\sin \phi \frac{J_1(kX \sin \theta)}{kX \sin \theta} \right).$$
(3)

To simplify things further, we choose $k = 2\pi$ and set

$$F(X; \theta = 15^{\circ}) = 10 \log_{10} \left| \frac{rE(X; r, \theta, 90^{\circ})}{E_0} \right|^2.$$
(4)

Thus, we let the radius be a stochastic variable $X \in U(2.5 \text{ m}, 3.5 \text{ m})$ and consider the quantity $\frac{rE}{E_0}$, which is *r*-independent, in dB in the E-plane at $\theta = 15^{\circ}$.

This example is quite simple, with an analytical F(X). We will use this example to illustrate the difference in convergence between a stochastic collocation method for UQ, such as that described in Section II-A, with a simple Monte-Carlo (MC) implementation.

To perform the comparison, a reference result for the mean μ_{ref} is produced by evaluating the function F(X) an extreme amount of times. Based on these reference values, the comparison shown in Figure 2 shows the absolute deviation between the reference result and the computed means μ_{calc} produced by the UQ algorithm and a simple MC implementation. Noting the log-log scale, we see that there is simply no reason to apply MC in a scenario such as those discussed in the present paper, where the number of variables is fairly low.

B. Horn Design

As a second example, we consider the design of an axially corrugated horn designed to illuminate a D = 1 m reflector in an offset configuration. The operating frequency is f = 12 GHz, and the focal length is F = 0.6 m. The geometry of the horn itself is shown in Figure 3, while the antenna system is shown in Figure 4.



Fig. 3. The corrugated horn used in Section III-B. The horn has 9 corrugations.

 TABLE I

 Key performance metrics of the system in Figure 4.

Quantity	Mean	Std. Dev	95% conf
Return loss	24.20	0.36	[23.48, 24.91]
Peak Directivity	40.06	0.02	[40.02, 40.09]
Cx-polar level	19.61	0.04	[19.52, 19.69]

The horn is rotationally symmetric and has 9 corrugations. It has been analyzed using an efficient Higher-Order Body of Revolution Method of Moments (HO-BoR-MoM) and optimized with a two-term objective function, meaning the horn provides a compromise between low return loss of the horn and high directivity of the reflector system. The reflector is analyzed using Physical Optics (PO) augmented with Physical Theory of Diffraction (PTD).

Based on the optimized design, we then add manufacturing and mounting deviations as follows:

- Ridge Widths, uniformly distributed ± 0.1 mm, 9 variables,
- Slot Widths, uniformly distributed ± 0.1 mm, 9 variables,
- Angular mounting angles, uniformly distributed $\pm 0.1^\circ, 2$ variables,
- Horizontal mounting in the focal plane, uniformly distributed ± 0.5 mm, 2 variables,

for a total of D = 22 variables.

The UQ algorithm requires 66 evaluations of $F(\overline{X})$ to produce the mean and standard deviation with sufficient accuracy, with a total computation time of 5 minutes on a laptop. This means that the analysis time for each evaluation of the complete reflector system is 4.5 seconds. The mean and confidence interval for the radiated pattern from the system is shown in Figure 5, while some of the key quantities of performance of the full system are shown in Table I.

Interestingly, while Table I shows that the return loss of the system is affected by the variation in the parameters, the patterns in Figure 5 demonstrate that the confidence interval of the radiated pattern is fairly narrow. This illustrates the importance of considering uncertainty in the entire system,



Fig. 4. An offset antenna system used in Section III-B, with the horn from Figure 3 used to illuminate the reflector.



Fig. 5. The co-polar and cx-polar patterns from the antenna system in Figure 4 for $\phi = 45^{\circ}$, with confidence intervals.

not just in the subcomponents individually, when performing uncertainty quantification. Thus, even though the variation in performance for the horn itself might be significant, this variation might be insignificant when the horn is used to illuminate a reflector.

C. Surface deviations for a simple reflector antenna design

We now move to consider an example of quantification of uncertainty stemming from inaccurate surfaces.



Fig. 6. Pattern radiated by the simple reflector system discussed in Section III-C when subjected to uncertainty in the manufactured surface.

We consider a very simple system—a center-fed D = 0.5 m parabolic reflector with F/D = 1.2 at f = 30 GHz. The simplicity of the design allows us to focus on the performance.

While the fundamental shape of the surface is a paraboloid, we add deviations in a 30-by-30 grid of points placed equidistantly on the surface, using interpolation between those 900 points. Each of the 900 points are then chosen to be stochastic variables with a uniform variation between -50 and 50 micron, intended to mimic deviations from the nominal paraboloidal reflector surface, caused by manufacturing errors.

The analysis is performed by PO/PTD, and requires 2700 evaluations of the $F(\overline{X})$ function. The total runtime is 6 minutes on a laptop, meaning that each evaluation runs in about 0.13 seconds. The resulting mean radiated pattern, along with a 95% confidence interval, is shown in Figure 6. We note that while the main beam is almost unaffected, the sidelobes shown a significant variation, with the confidence interval at the peak of the first sidelobe being about ± 1 dB. Thus, if the sidelobe levels are important for the application, perhaps ± 50 micron is an unacceptable deviation for this design.

D. Multi-beam

As a final example, we will consider a scenario based on a High-Throughput-Satellite (HTS) reflector antenna configuration, where the individual feeds in a feed cluster each use the reflector to produce seperate beams, i.e., a multi-beam single-feed per beam setup. The system configuration is a standard four reflector setup, where four reflectors are needed to produce the required number of beams. Each reflector uses multiple offset feeds to produce a number of beams with the same polarization and frequency. The beams of the system are shown in Figure 7.

Each reflector uses a single offset reflector with D = 3.66 m at F = 30 GHz. The beam-width is 0.25° . In such systems, it is important that the sidelobes of one beam in the nearest neighbouring coverage cell with the same frequency and



Fig. 7. The beams from a High-Throughput Satellite as discussed in Section III-D. Beams with the same color have the same polarization and frequency and are generated by offset feeds using the same reflector.

polarisation are not too high, to avoid compromising the C/I (Carrier to interference) ratio. In our system, this means that the most interesting cell to look at begins 0.375° away from the peak.

Therefore, in Figure 8, we have performed two separate uncertainty quantifications, examining the beams from two coverage cells with the same color. The critical factor for these two beams is thus the co-polar level of the sidelobes interfering with the other beam.

Similar to the previous case, we add deviations to a paraboloidal surface in a 20-by-20 grid on the surface using interpolation between those 400 points. We let those surface deviations vary uniformly between -100 and 100 micron, assuming that the significantly larger reflector will be less accurate than the previous case. The simulation requires 1201 evaluations of the system, and completes in roughly 25 minutes.

The results from Figure 8 are clear. While the performance of the system with the nominal surface might be acceptable, the surface variation significantly affects the sidelobe levels. In turn, the performance of the system as measured by C/I could very well be significantly poorer than indicated by the nominal simulation, when the surface is actually produced.

IV. CONCLUSION

This paper has highlighted the capabilities of uncertainty quantification in the practical setting of reflector antenna design, namely, to achieve a much more realistic estimate of the performance of the system when it has been manufactured, rather than considering a canonical and simplified geometry that is unaffected by production errors. The results clearly demonstrate that a higher-order uncertainty quantification algorithm is able to achieve considerably better accuracy



Fig. 8. Pattern from two feeds with the same frequency and polarization in a standard four-color setup, as described in Section III-D. Thus, the sidelobe level from one beam in the coverage (indicated by the yellow region) of the other beam will affect the C/I of the system.

than Monte-Carlo methods in far fewer function evaluations, allowing a more realistic answer to be achieved in a shorter timeframe. The take-away conclusion is thus obvious: Modern antenna designers should ensure that their computational tools allow them to quantify the uncertainty of their designs using efficient and accurate algorithms. This will allow antenna engineers to simulate what they can actually expect to measure, instead of simulating an idealised performance that can rarely, if ever, be realized in practice.

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REFERENCES

- D. Xiu and J. S. Hesthaven, "High-Order Collocation Methods for Differential Equations with Random Inputs," *SIAM Journal on Scientific Computing*, vol. 27, no. 3, pp. 1118–1139, Jan. 2005.
- [2] L. Ng and M. S. Eldred, "Multifidelity uncertainty quantification using nonintrusive polynomial chaos and stochastic collocation," in AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, 2012.
- [3] G. W. Wasilkowski and H. Wozniakowski, "Explicit cost bounds of algorithms for multivariate tensor product problems," *Journal of Complexity*, 1995.
- [4] A. H. Stroud, "Remarks on the disposition of points in numerical integration formulas," *Mathematical Tables and Other Aids to Computation*, 1957.
- [5] C. A. Balanis, Advanced Engineering Electromagnetics, 2nd Edition. Wiley Global Education, Jan. 2012.