Gaussian Translation Operator for Multi-Level Fast Multipole Method

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***Abstract*—Results using a new translation operator for the Multi-Level Fast Multipole Method are presented. Based on Gaussian beams, the translation operator allows a significant portion of the plane-wave directions to be neglected, resulting in a much faster translation step.**

I. Introduction

The Multi-Level Fast Multipole Method (MLFMM) is a popular method for reducing the memory and computational complexity of solving the electromagnetic scattering problem. Based on a Galerkin discretization, MLFMM achieves reduced complexity by grouping the basis functions hierarchically and letting larger groups interact over greater distances.

The MLFMM represents the matrix vector product $\overline{\overline{Z}}\overline{I}$ as

$$\overline{\overline{Z}}\overline{I} = \overline{\overline{Z}}_{\text{near}}\overline{I} + \mathcal{F}(\overline{I}) \tag{1}$$

where $\overline{\overline{Z}}_{\text{near}}$ is the near-matrix, containing the interactions between basis functions that are too closely spaced to apply MLFMM. The elements in this matrix are computed as in the normal Galerkin approach. $\mathcal{F}$ denotes the operation perfomed by applying MLFMM.

The interaction between two well-seperated basis functions $\boldsymbol{f}_j, \boldsymbol{f}_i$, belonging to groups m and m' respectively, can be computed by

$$\overline{\overline{Z}}_{j,i} = \kappa \oiint \boldsymbol{R}_{jm}(k,\hat{\boldsymbol{k}}) \cdot \Big(T_L(k,\hat{\boldsymbol{k}},\boldsymbol{r}_{mm'})\boldsymbol{V}_{im'}(k,\hat{\boldsymbol{k}})\Big)\, d^2\hat{\boldsymbol{k}}, \tag{2}$$

with $\boldsymbol{r}_{mm'} = \boldsymbol{r}_m - \boldsymbol{r}_{m'}$, where $\boldsymbol{r}_m$ is the center of group m, and for the Electric Field Integral Equation (EFIE) $\kappa = i\frac{k\eta}{4\pi}$, where η is the free-space impedance. For EFIE, the basis function signature $\boldsymbol{R}_{jm}(k,\hat{\boldsymbol{k}}_p) = \boldsymbol{V}^*_{jm}(k,\hat{\boldsymbol{k}}_p)$, where * denotes complex conjugation, and

$$\boldsymbol{V}_{jm}(k,\hat{\boldsymbol{k}}) = \int_{\boldsymbol{r}^2} \boldsymbol{f}_j(\boldsymbol{r}) \cdot [\overline{\overline{I}} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}] e^{ik\hat{\boldsymbol{k}}\cdot(\boldsymbol{r}_m - \boldsymbol{r})}\, d^2\boldsymbol{r}, \tag{3}$$

and Rokhlins translation function T_L [1] is computed as

$$T_L(k,\hat{\boldsymbol{k}},\boldsymbol{x}) = \sum_{l=0}^{L} i^l(2l+1)h_l^{(1)}(k|\boldsymbol{x}|)P_l(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{x}}). \tag{4}$$

Herein, $\hat{\boldsymbol{k}}$ is the unit wave vector, $\boldsymbol{x}$ is the vector between two group centers directed towards the receiving group, $\hat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$, $h_l^{(1)}$ is the spherical Hankel function of the first kind and order l, and P_l is the Legendre polynomial of order l. The number of distinct plane wave directions is $K_L = 2(L+1)^2$, and L is typically chosen as

$$L = kD + 1.8\beta^{2/3}(kD)^{1/3} \tag{5}$$

where D is the group diameter and the relative error is $10^{-\beta}$.

For each matrix vector product, the translation operator T_L is performed on every pair of interacting groups. Thus, the translation operator is a significant contribution to the overall matrix vector multiplication time. Therefore, there has been significant effort into developing a translation operator that discards some of the plane-wave directions [2]–[5]. The present paper uses the Gaussian translation operator from [5], which comes with a sampling theorem that guarantees arbitrary accuracy, all the way to machine precision.

II. Gaussian Translation Operator

Adding an imaginary vector to the source point and subtracting this same imaginary vector from the origin of the source coordinate system, such that the real displacement between source and receiver becomes

$$\boldsymbol{r}_1 + \boldsymbol{r}_{mm'} + \boldsymbol{r}_2 = (\boldsymbol{r}_1 + \boldsymbol{r}_2 - i\Delta\hat{\boldsymbol{r}}_{mm'}) + \hat{\boldsymbol{r}}_{mm'}(|\boldsymbol{r}_{mm'}| + i\Delta), \tag{6}$$

where $\boldsymbol{r}_1 = \boldsymbol{r}_m - \boldsymbol{r}$ and $\boldsymbol{r}_2 = \boldsymbol{r}' - \boldsymbol{r}_{m'}$, the translation operator becomes [5]

$$T_N(k,\hat{\boldsymbol{k}},\boldsymbol{x},\Delta) = e^{k\Delta(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{x}}-1)} \sum_{n=0}^{N} i^n(2n+1)\tilde{h}_n^{(1)}(k[|\boldsymbol{x}| + i\Delta])P_n(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{x}}), \tag{7}$$

with $\tilde{h}_n^{(1)}(x) = h_n^{(1)}(x)e^{\text{Im}(x)}$ being the normalized spherical Hankel function. This translator is based on a point source residing in the complex domain — such a point source has a far-field pattern which locally has Gaussian behaviour, hence the name Gaussian translation operator.

By increasing Δ, the error for a fixed upper limit N becomes larger, but the translation operator T_N becomes sharper. Consequently, only a small subset of the tabulated directions are necessary, and the rest can be discarded during the translation stage. To allow for a larger value of Δ whilst not reducing the accuracy, N has to be increased relative to the usual upper limit L chosen for the Rokhlin translator. Furthermore, due to numerical concerns regarding cancellation, there is an upper limit to how large the term $\Delta/|\boldsymbol{x}|$ can be.

Thus, to apply the Gaussian translation operator, one needs to choose allowed upper limits for N and $\Delta/|\boldsymbol{x}|$. Based on this, a value of Δ can be found by increasing Δ until either $\frac{\Delta}{|\boldsymbol{x}|}$ is greater than the limit, or until

$$\left| \frac{e^{ik|\boldsymbol{r}_e+\boldsymbol{x}|}/|\boldsymbol{r}_e+\boldsymbol{x}| - \sum_{n=0}^{N} U_n(k,\boldsymbol{x},\boldsymbol{b},\Delta)}{e^{ik|\boldsymbol{r}_e+\boldsymbol{x}|}/|\boldsymbol{r}_e+\boldsymbol{x}|} \right| \approx 10^{-\beta} \quad (8)$$

where $\boldsymbol{b} = \boldsymbol{r}_e - i\Delta\hat{\boldsymbol{x}}$, and U_n is [5]

$$\begin{aligned} U_n(k,\boldsymbol{x},\boldsymbol{b},\Delta) =& ik(-1)^n(2n+1) \quad (9)\\ & \cdot h_n^{(1)}(k[|\boldsymbol{x}|+i\Delta])j_n(k\sqrt{\boldsymbol{b}\cdot\boldsymbol{b}})P_n\left(\frac{\hat{\boldsymbol{x}}\cdot\boldsymbol{b}}{\sqrt{\boldsymbol{b}\cdot\boldsymbol{b}}}\right), \end{aligned}$$

where j_n is the spherical Bessel function. We set $\boldsymbol{r}_e = D\boldsymbol{a}$, where D is the diameter of the groups interacting and $\boldsymbol{a}$ is an otherwise arbitrary vector orthogonal to $\boldsymbol{x}$. Through numerical testing, we have found the upper limit on $\frac{\Delta}{|\boldsymbol{x}|}$ to be 0.15. While this will depend on the required accuracy, and the location of sources inside the interacting groups [6], we have found 0.15 to be a reasonable rule of thumb.

III. Implementation Details

Having found Δ for each translator $T_N(k,\hat{\boldsymbol{k}},\boldsymbol{r}_{mm'},\Delta)$, the translator is evaluated and only the tabulated plane-wave directions $\hat{\boldsymbol{k}}_t$ for which

$$|T_N(k,\hat{\boldsymbol{k}}_t,\boldsymbol{r}_{mm'},\Delta)| > \max_{\hat{\boldsymbol{k}}} |T_N(k,\hat{\boldsymbol{k}},\boldsymbol{r}_{mm'},\Delta)| \cdot 10^{-(\beta+2)} \quad (10)$$

are included in the translation stage. Since the sharpness of the translator is directly proportional to the value of Δ, this demonstrates why increasing Δ as much as possible is desirable. Thus, there is a comprimise between increasing Δ as much as possible, to increase computational speed, and keeping N and thus $K_N = 2(N+1)^2$ as low as possible, to reduce memory costs for the group patterns. The number of included plane-wave directions is labelled Q in the following.

We do not utilize the Gaussian translator if $Q/K_N > 0.8$, since there is some computational overhead involved in keeping track of the directions to be skipped by the Gaussian translator.

IV. Numerical Results

As a testcase, we consider the scattering from a 100λ square PEC plate illuminated by a plane wave. The implementation used is discussed in detail in [7]. We use $N = L + 5$ for all levels save for the finest, and set $\beta = 2$. Comparing with the usual Rokhlin translator, the results are shown in Figure 1, while the key numbers for the translator are shown in Table I.

From the figure, we see that the resulting scattered fields are effectively the same, with a relative deviation of less than 0.2%. The truly interesting factor, however, is the reduction in computational time as seen in the table — the translation time is reduced by almost 40%, which is very significant in many implementations since the translation stage occupies a significant portion of the time. Further, while the increased N compared to L results in slightly larger memory requirements

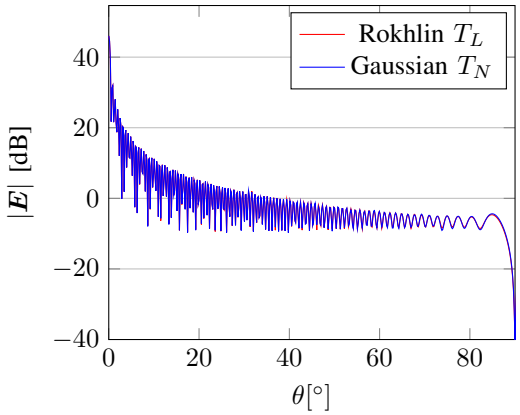

Fig. 1. Scattered co-polar fields in the E-plane calculated using T_L and T_N, respectively. The relative deviation is less than 0.2%.

TABLE I
Savings in computational time and memory of the Gaussian operator relative to the Rokhlin operator.

	Q/K_N	Gaussian (T_N) relative to Rokhlin (T_L)		
		Time	Group memory	Translator memory
Level 3	0.23	0.55	1.03	0.23
Level 4	0.30	0.53	1.06	0.32
Level 5	0.40	0.79	1.12	0.45
Level 6	0.81	1.00	1.00	1.00
Total	–	0.64	1.07	0.26

for the group patterns, the translators themselves require much less memory. The latter feature is important for distributed memory implementations, where the translators are often replicated on each node to reduce inter-node communication.

V. Conclusion

The performance of a Gaussian translation operator, used with an actual scatterer, has been presented for the first time, demonstrating a significantly reduced matrix-vector product time compared to the Rokhlin translator. The implementation is fairly simple and only requires modifying the parts of the code involved in translation. The potential for other MLFMM implementations, such as distributed memory or on-the-fly evaluation of translators, is obvious.

References

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