# Caustics and Caustic Corrections to the Field Diffracted by a Curved Edge

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*Abstract*—**The caustics for the singly diffracted rays from a circular disk are investigated when the rays emanate from a source which is placed off the axis of the disk. Either two or four singly diffracted rays appear dependent on the far-field direction. The caustics separate the region of space with two singly diffracted rays from the regions with four singly diffracted rays. Corrections are derived which allow the ray description to be continued across the caustics. These corrections depend on the local properties of the edge at the point of diffraction and may be used for edges with arbitrary curvature. Numerical examples are included to demonstrate how the caustic corrections improve calculations based on the geometrical theory of diffraction (GTD).**

## I. INTRODUCTION

IT IS WELL KNOWN that the geometrical theory of diffraction (GTD), Keller [1], leads to singular expressions for the field in certain directions, e.g., at shadow and reflection boundaries and at caustics. The singularities at shadow and reflection boundaries may be avoided through the introduction of uniform diffraction coefficients, Ahluwalia *et al.* [2] and Kouyoumjian and Pathak [3], while the singularities at caustics may be removed by the introduction of caustic corrections. For an axial caustic a correction factor has been derived by Keller [4]. In the present paper we consider the case of nonaxial caustics.

For a source on the axis of a circular disk, the diffracted field is focused in the axial directions and the axial caustics appear. If the source is moved away from the axis, the caustics move in the opposite direction away from the axis and defocusing takes place. Now the caustics for the GTD field form two diamond-shaped figures on the far-field sphere. As the displacement of the source away from the axis is increased, the size of the two sectors enclosed within the caustics also increase and at a certain stage they coalesce. Inside the diamond-shaped figures, four singly diffracted rays contribute to the field. Outside the figures, only two singly diffracted rays contribute. The caustic corrections derived in this paper extend GTD to be valid across a caustic and connect the singly diffracted fields on the two sides of a caustic. Although the corrections are derived for a circular disk, they are of a general nature and correct the GTD field for singularities caused by the curvature of a diffracting edge.

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The caustic corrections are obtained by means of the principle of equivalent edge currents [5], [6] and asymptotic expressions for phase integrals with two or three interacting stationary points [7]. For sources close to the axis of a circular disk, the stationary points are not distinct and the radiation from the edge cannot be attributed to discrete diffraction points. In this case, the diffracted field may be obtained by integrating the edge currents either numerically or analytically [4], [8]. The integral representation of the diffracted field and its asymptotic evaluation for isolated stationary points are reviewed in Section II. This asymptotic evaluation is not valid near the caustics where the diffraction integral possesses two or three stationary points close to each other. In Section III we consider examples of caustics for the circular disk.

Caustic corrections to the GTD field are derived in Section IV for the case where the field point is close to a caustic but well away from a cusp of caustics. Corrections valid near a cusp of caustics are given in Section V. The numerical examples in Section VI demonstrate how the caustic corrections are utilized to improve GTD calculations.

## II. DIFFRACTION IN A CIRCULAR DISK

In order to examine the field diffracted in a curved edge we consider the circular disk with radius $a$ in Fig. 1. The disk is illuminated by a point source $S$ in the first quadrant of the $zx$ plane. The equivalent currents give the following contribution to the electric far-field:

$$\bar{E}(\theta, \phi) = \frac{e^{ikr}}{r} \int_0^{2\pi} \bar{G}(\phi') e^{ikh(\phi')}\, d\phi' \tag{1}$$

where $k$ is the wavenumber. The time factor $e^{-i\omega t}$ has been suppressed. This contribution accounts for the singly diffracted field in the direction defined by the angles $\theta$ and $\phi$. The variable of integration $\phi'$ is the angle to a point $D$ on the edge. The $\theta$ and $\phi$ components of the function $\bar{G}(\phi')$ are given by

$$\begin{Bmatrix} G_\theta \\ G_\phi \end{Bmatrix} = \sqrt{\frac{k}{2\pi}}\, e^{i(3\pi/4 - kR(\phi'))} \cdot \begin{Bmatrix} \cos\theta \sin(\phi-\phi') & \cos(\phi-\phi') \\ \cos(\phi-\phi') & -\cos\theta\sin(\phi-\phi') \end{Bmatrix} \begin{Bmatrix} E_\beta{}^i D_s \\ E_\phi{}^i D_h \end{Bmatrix}. \tag{2}$$

Here $E_\beta{}^i$ and $E_\phi{}^i$ are the components of the incident electric field at $D$ in a ray fixed coordinate system and $D_s$ and $D_h$ are

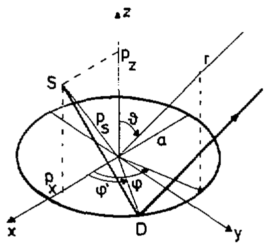


Fig. 1. Source above circular disk.

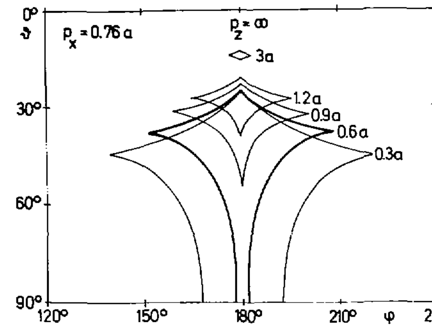


Fig. 2. Caustics of diffracted field for source placed at fixed distance, $p_x = 0.76a$, from axis of circular disk. Height $p_z$ of source above disk is parameter.

uniform diffraction coefficients for the soft and the hard boundary condition [3].

With the distance $r$ to the field point referred to the origin, the phase function $h(\phi')$ is given by

$$h(\phi') = R(\phi') - a \sin \theta \cos (\phi - \phi') \tag{3}$$

where

$$R(\phi') = \sqrt{p_s^2 + a^2 - 2p_x\, a \cos \phi'} \tag{4}$$

is the distance from the source point $S$ to the point of integration $D$ on the edge.

When the far-field direction is away from the caustics, the phase function (3) has two or four first-order stationary points $\phi_0'$ where $h'(\phi_0') = 0$. (The prime denotes differentiation with respect to the angle $\phi'$.) Each of these stationary points gives an asymptotic contribution

$$\bar{E}' \sim \sqrt{\frac{2\pi}{k\,|h''(\phi_0')|}}\, \bar{G}(\phi_0') e^{ikh(\phi_0') \pm i\pi/4} \frac{e^{ikr}}{r} \tag{5}$$

to the field. The plus sign applies for $h''(\phi_0') > 0$ and the minus sign for $h''(\phi_0') < 0$ [7, p. 382]. Equation (5) could also be obtained by a direct application of GTD. In the derivation of (5) from (1) we assume that the amplitude function $\bar{G}(\phi')$ is regular and slowly varying in the vicinity of the stationary point $\phi_0'$. This assumption applies throughout the paper.

## III. OFF-AXIS CAUSTICS FOR A CIRCULAR DISK

For $h''(\theta, \phi; \phi_0')$ equal to zero, (5) is singular, and the far-field direction $(\theta, \phi)$ is a caustic direction for the field diffracted at $\phi_0'$. Thus, the caustic directions corresponding to $\phi_0'$ are obtained when the two equations $h'(\theta, \phi; \phi_0') = 0$ and $h''(\theta, \phi; \phi_0') = 0$ are solved simultaneously. These directions form diamond-shaped figures on the far-field sphere as shown in Fig. 2. This figure also shows the variation of the caustic curves when the height of the source above the disk is varied while the distance between the axis and the source is kept constant at approximately three quarters of the radius of the disk. The caustics occur in the half sphere opposite the source and may extend over a considerable part of the far-field sphere.

The concept of a caustic or ray envelope is illustrated in Fig. 3 which shows a two-dimensional ray picture near a cusp of caustics. Three rays contribute to the field at point $A$ between the two caustics. Ray 1 is a tangent ray to caustic $a$ while rays 2 and 3 are tangent rays to caustic $b$. Ray 2 grazes caustic $b$ after passing the field point while ray 3 grazes the caustic before it reaches the field point. If the field point is moved along the dotted line toward $B$ on caustic $b$, rays 2 and 3 approach each other. Rays 2' and 3' show rays 2 and 3 just before the field point reaches the caustic. Close to the caustic, the two rays cannot be dealt with separately. Close to the cusp of caustics all three rays merge.

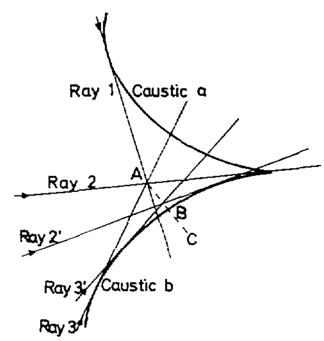


Fig. 3. Rays near cusp of caustics.

## IV. CAUSTIC CORRECTIONS AWAY FROM A CUSP OF CAUSTICS

In this section we consider the field due to the rays 2 and 3 in Fig. 3 as the field point moves from $A$ on the lit side across the caustic at $B$ to $C$ on the shadow side. The crossing of the caustic takes place far away from the cusp so that ray 1 is not affected by rays 2 and 3.

Close to the lit side of a caustic, the second derivative of the phase function in (1) is small, and the GTD ray description is not valid. The phase function may then be expanded into

$$h(\phi') \cong h(\phi_2') + \tfrac{1}{2}h''(\phi_2')(\phi' - \phi_2')^2 + \tfrac{1}{6}h'''(\phi_2')(\phi' - \phi_2')^3 \tag{6}$$

where $\phi_2'$ is the stationary point corresponding to ray 2. The stationary point $\phi_3'$ corresponding to ray 3 is in (6) displaced $-2h''(\phi_2')/h'''(\phi_2')$ away from $\phi_2'$. When (6) is substituted into (1) and $\bar{G}(\phi')$ is replaced by its value at the stationary point, the resulting phase integral may be transformed into the

Airy function as shown in Appendix A. The contribution due to rays 2 and 3 becomes

$$\bar{E} \sim 2\pi \left( \frac{2}{kh'''(\phi_2')} \right)^{1/3} \bar{G}(\phi_2') \operatorname{Ai}(-\sigma) \cdot e^{i\frac{2}{3}\sigma^{3/2}} e^{ikh(\phi_2')} \frac{e^{ikr}}{r} \tag{7}$$

where the argument of the Airy function

$$-\sigma = -\left(\frac{k}{2}\right)^{2/3} \frac{h''(\phi_2')^2}{h'''(\phi_2')^{4/3}} \tag{8}$$

is negative. The parameter $\sigma$ is a measure of the distance to the caustic. Thus, $\sigma \to 0$ corresponds to the case where the two rays coalesce at the caustic into one contribution due to a second-order stationary point, whereas $\sigma \to \infty$ corresponds to the conventional GTD solution away from the caustic where the two rays exist independently.

The Airy function is asymptotically equal to

$$\operatorname{Ai}(-\sigma) \sim \frac{1}{\sqrt{\pi}\sigma^{1/4}} \sin\left(\tfrac{2}{3}\sigma^{3/2} + \pi/4\right)$$

$$= \frac{1}{2\sqrt{\pi}\sigma^{1/4}} \left(e^{i\frac{2}{3}\sigma^{3/2} - i\pi/4} + e^{-i\frac{2}{3}\sigma^{3/2} + i\pi/4}\right) \tag{9}$$

for $\sigma \to \infty$.

Substitution of (9) into (7) shows that (7) takes into account both ray 2 and ray 3. Away from the caustic, (7) tends towards a sum of two terms, each of which is similar to (5). It is suggested, therefore, that each GTD ray contribution (5) be multiplied by the caustic correction factor

$$C = \frac{\operatorname{Ai}(-\sigma)\sqrt{\pi}\sigma^{1/4}}{\sin\left(\tfrac{2}{3}\sigma^{3/2} + \pi/4\right)} \tag{10}$$

where $\sigma$ is defined by (8). Then the sum of the corrected contributions for ray 2 and ray 3 attains the required limiting value at the caustic.

The derivatives appearing in the caustic distance parameter $\sigma$ (8) should be evaluated at the stationary point which corresponds to the ray under consideration. For a circular disk, these derivatives are easily obtained from (3). In the general case, the caustic correction factor will depend on the local properties of the diffracting edge at the point of diffraction.

The caustic correction factor (10) removes the singularity of the GTD field at a caustic. As (10) is defined as the ratio of the Airy function to its asymptotic expression for $\sigma$ large, it becomes equal to unity far away from the caustic. In practice this occurs for $\sigma$ equal to about 1.2.

On the shadow side of the caustic, from $B$ towards $C$, the rays 2 and 3 do not appear. This corresponds to the situation at point $C$ in Fig. 3 where rays 2 and 3 do not contribute to the field. In this case, the expansion of the phase function is

$$h(\phi') \cong h(\phi_0') + h'(\phi_0')(\phi' - \phi_0') + \tfrac{1}{6}h'''(\phi_0')(\phi' - \phi_0')^3 \tag{11}$$

[9, p. 484]. The angle $\phi_0'$ in (11) is obtained numerically as a zero of $h''(\phi')$. This zero is a reminiscence of the second-order stationary point on the caustic. The zero should be chosen so that $h'(\phi_0')$ and $h'''(\phi_0')$ have the same sign.

When (11) is substituted into the integral representation (1) of the diffracted field, the following formula ensues:

$$\bar{E} \sim 2\pi \left( \frac{2}{kh'''(\phi_0')} \right)^{1/3} \bar{G}(\phi_0') \operatorname{Ai}(\sigma) e^{ikh(\phi_0')} \frac{e^{ikr}}{r} \tag{12}$$

where the parameter $\sigma$ for the distance to the caustic is given by

$$\sigma = kh'(\phi_0') \left( \frac{2}{kh'''(\phi_0')} \right)^{1/3} \tag{13}$$

When $h'(\phi_0')$ and $h'''(\phi_0')$ have the same sign, the argument of the Airy function is positive. Then the Airy function is asymptotically given by

$$\operatorname{Ai}(\sigma) \sim \frac{1}{2\sqrt{\pi}\sigma^{1/4}} e^{-\frac{2}{3}\sigma^{3/2}} \tag{14}$$

for $\sigma \to \infty$. Thus (12) is a contribution which in a region with only two singly diffracted rays from a disk may be added to the field. This contribution decreases exponentially away from the caustic and accounts for the two rays which disappear at the caustic. It will be referred to as the caustic shadow term.

## V. CAUSTIC CORRECTIONS NEAR A CUSP OF CAUSTICS

At a cusp of caustics, the phase function in (1) possesses a third-order stationary point. Then $h'''(\phi')$ is equal to zero, and the caustic corrections (10) and (12) are not valid. In this case, expansion of the phase function about the stationary point $\phi_2'$ of the central ray, ray 2 in Fig. 3, yields

$$h(\phi') \cong h(\phi_2') + \tfrac{1}{2}h''(\phi_2')(\phi' - \phi_2')^2 + \tfrac{1}{24}h^{(4)}(\phi_2')(\phi' - \phi_2')^4. \tag{15}$$

When $h''(\phi_2')$ and $h^{(4)}(\phi_2')$ have opposite signs, (15) gives three equispaced stationary points which coalesce for $h''(\phi_2')$ equal to zero. When $h''(\phi_2')$ and $h^{(4)}(\phi_2')$ have the same sign, two of the stationary points are imaginary and only the central stationary point is important. For a circular disk, the case with only one important stationary point occurs when the field point is in the region with two singly diffracted rays.

Substitution of (15) into the phase integral

$$I = e^{-ikh(\phi_2')} \int_0^{2\pi} e^{ikh(\phi')}\, d\phi' \tag{16}$$

and transformation into the parabolic cylinder function yields for $h^{(4)}(\phi_2')$ positive

$$I \sim \left(\frac{12}{kh^{(4)}(\phi_2')}\right)^{1/4} \sqrt{\pi} D_{-\frac{1}{2}}\left(h''(\phi_2')\sqrt{\frac{3k}{h^{(4)}(\phi_2')}} \cdot e^{-i\pi/4}\right) e^{-i\frac{3}{4}kh''(\phi_2')^2/h^{(4)}(\phi_2')+i\pi/8} \tag{17}$$

as shown in Appendix A.

A similar expression is given in [7, p. 812] for the field above a uniaxially anisotropic plasma half-space where a magnetic line current is embedded. Use of the large argument approximations to the parabolic cylinder function gives for $h''(\phi_2')$ positive

$$I \sim \sqrt{\frac{2\pi}{kh''(\phi_2')}}\, e^{i\pi/4} \tag{18}$$

and for $h''(\phi_2')$ negative

$$I \sim \sqrt{\frac{-2\pi}{kh''(\phi_2')}}\,\Big(e^{-i\pi/4} + \sqrt{2}\, e^{-i\frac{3}{2}kh''(\phi_2')^2/h^{(4)}(\phi_2')+i\pi/4}\Big). \tag{19}$$

In (18) only one stationary point contributes, and the field point is in a region with only two singly diffracted rays from a circular disk. When $h''(\phi_2')$ and $h^{(4)}(\phi_2')$ have opposite signs as in (19), all three stationary points contribute. This corresponds to a case where the disk gives rise to four singly diffracted rays. The first term of (19) corresponds to the central ray, ray 2 in Fig. 3, while the second term gives the two identical contributions from ray 1 and 3. The contribution from each of these rays is $\sqrt{2}$ times smaller than the contribution from ray 2 because

$$h''(\phi_1') \cong h''(\phi_3') \cong -2h''(\phi_2') \tag{20}$$

[7, p. 812]. The value of $h^{(4)}(\phi')$ is approximately the same for the three rays. Thus it is important to know if a ray is a noncentral ray since for such a ray $h''(\phi')$ must be changed according to (20). Noncentral rays occur in regions where for the disk four singly diffracted rays are present. Noncentral rays are easily recognized because for these rays the signs of $h''(\phi')$ and $h^{(4)}(\phi')$ are equal.

Near a cusp of caustics, caustic correction factors are obtained as the ratio of (17) to either (18) or (19) dependent on whether the field point is in a region with two or four singly diffracted rays. For $h^{(4)}(\phi_2')$ positive, the caustic correction factor is

$$C = \left(\frac{3k}{h^{(4)}}\right)^{1/4} \sqrt{h''} \cdot \frac{D_{-\frac{1}{2}}\left(h''\sqrt{\frac{3k}{h^{(4)}}}\, e^{i3\pi/4}\right)}{e^{i\frac{3}{4}kh''^2/h^{(4)}-i\pi\frac{3}{8}} + \sqrt{2}\, e^{-i\frac{3}{4}kh''^2/h^{(4)}+i\pi/8}} \tag{21}$$

in a region with four stationary points and

$$C = \left(\frac{3k}{h^{(4)}}\right)^{1/4} \sqrt{h''} \cdot \frac{D_{-\frac{1}{2}}\left(h''\sqrt{\frac{3k}{h^{(4)}}}\, e^{-i\pi/4}\right)}{e^{i\frac{3}{4}kh''^2/h^{(4)}+i\pi/8}} \tag{22}$$

in the region with two stationary points. Here $h''$ and $h^{(4)}$ are the absolute values of the second derivative and of the fourth derivative of the phase function for the central ray. If a noncentral ray is considered, the second derivative should be multiplied by $-½$ according to (20). For $h^{(4)}(\phi_2')$ negative, the complex conjugated expressions of (21) and (22) apply. For the cusp correction factors, the role of the caustic distance parameter $\sigma$ is taken by $h''\sqrt{3k/h^{(4)}}$. For this parameter larger than 2, the cusp correction factors may be approximated by unity.

## VI. APPLICATION OF CAUSTIC CORRECTIONS

So far we have not considered the transition from the Airy function corrections (10) and (12) to the cusp correction factors (21) and (22). A rigorously derived transition requires that the series expansion (15) of the phase function also include a third-order term. The phase integral for this general case has been evaluated numerically by Pearcey [10]. In order to maintain the speed of the GTD field calculations also in the vicinity of caustics, a less rigorous approach was adopted as described in the following. In a region with four singly diffracted rays from a disk, both the Airy correction factor (10) and the cusp correction factor (21) are evaluated for each ray and the GTD ray contribution is multiplied by the correction factor which has the smaller amplitude. As a result, the Airy function correction factor will be utilized when the field point is close to a caustic but away from a cusp of caustics. When the field point is close to a cusp of caustics, the cusp correction factor (21) is utilized.

If a caustic correction is required in a region with two singly diffracted rays from a disk, either the caustic shadow term (12) is added to the GTD field or one of the two singly diffracted rays is modified by the cusp correction factor (22). The caustic shadow term (12) exists only if $h''(\phi')$ possesses a zero, $\phi_0'$, where $h'(\phi_0')$ and $h'''(\phi_0')$ have the same sign. In this case the shadow term is used when

$$|kh'''(\phi_0')|^{1/3} > 0.7\,|kh^{(4)}(\phi_0')|^{1/4}. \tag{23}$$

Otherwise the cusp correction factor is used.

The computer program used in the following examples calculates the far-field from arbitrarily positioned sources near a finite circular cylinder. In addition to the direct and the reflected ray, the program takes into account the singly diffracted rays from each end surface and creeping waves. The diffracted rays are corrected for axial and nonaxial caustics and the creeping waves include penumbra region corrections. The nonaxial caustic corrections used are those described in

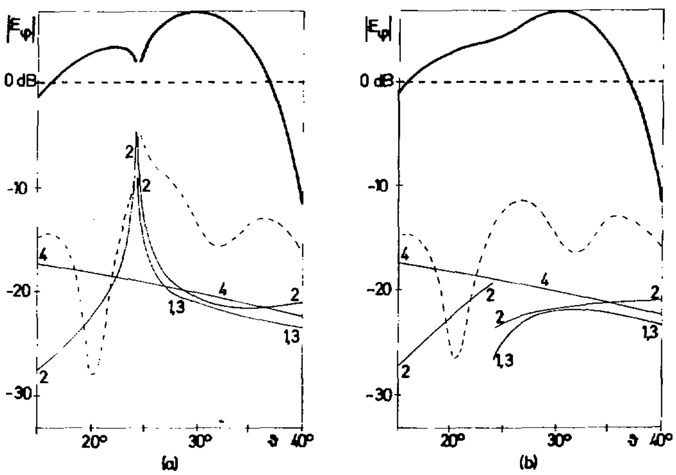


Fig. 4. GTD field across cusp of caustics. (a) Without caustic corrections. (b) With caustic corrections. Direct and reflected ray are of equal magnitude. — Total GTD field. ---- Direct and reflected rays. — Singly diffracted rays. ---- Sum of singly diffracted rays.

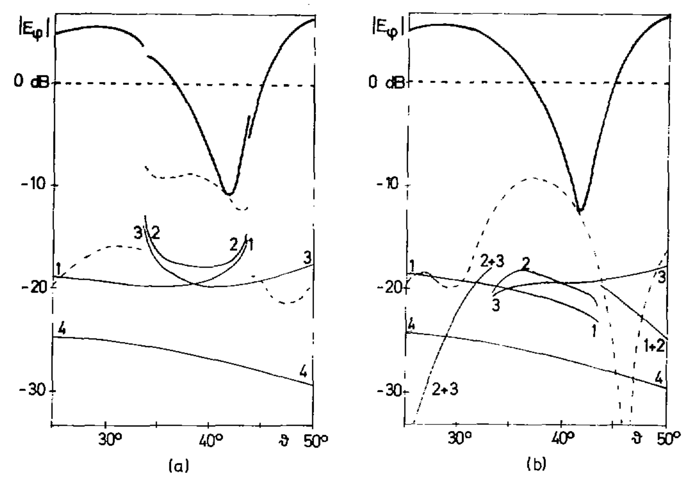


Fig. 5. GTD field across caustic away from cusp of caustics. (a) Without caustic corrections. (b) With caustic corrections. Direct and reflected ray are of equal magnitude. — Total GTD field. ----Direct and reflected rays. — Singly diffracted rays. ---- Sum of singly diffracted rays.

Sections IV and V. Details of the numerical evaluation of the corrections are given in Appendix B.

Figs. 4(a) and 4(b) show GTD fields without and with caustic corrections, respectively. The figures show how the field varies through a cusp at $\phi = 180^\circ$ as a function of the angle $\theta$. The field originates from a half-wave dipole placed above the end surface of a vertical, finite circular cylinder. The radius of the cylinder is 10 cm and the frequency 10 GHz. The dipole is parallel to the $y$ axis in Fig. 1 where, if the circle indicates the end surface of the cylinder, the position of the source is given by $p_x = 7.6$ cm and $p_z = 6$ cm. The resulting caustic is shown in thick line on Fig. 2. The direct ray, the reflected ray, the individual diffracted rays, the total diffracted field and the total field are shown. The diffracted rays are numbered according to Fig. 3 so that the central ray has number 2. The two identical noncentral rays are numbered 1 and 3. The ray numbered 4 is left unaffected by the passage of the cusp of caustics. The cusp correction factors modify the other three diffracted rays in such a way that the total diffracted field becomes continuous across the cusp.

Figs. 5(a) and 5(b) show, for the same configuration as in Fig. 4, the GTD field without and with caustic corrections, respectively, as a caustic is crossed well away from a cusp of caustics. The field is calculated for $\phi = 166^\circ$ as a function of $\theta$. The field point enters the caustic region at $\theta = 33^\circ$ and leaves it at $\theta = 44^\circ$. At $\theta = 33^\circ$ the caustic shadow term, indicated as $2 + 3$, builds up before the crossing of the caustic curve. Here ray 2 and 3 come into existence. These rays are modified by the Airy function correction factors (10) while rays 1 and 4 are not affected by the passage of the caustic. At $\theta = 44^\circ$ rays 1 and 2 disappear. Their disappearance is compensated for by the caustic shadow term $1 + 2$.

In the above examples, the direct and the reflected rays are large and the caustic appears to have only a minor effect on the total field except, of course, at the caustic itself where GTD predicts an infinite field. In regions where only the diffracted field is present, the influence of the caustics could be much more noticeable. If the patterns in Figs. 4 and 5 are continued until $\theta = 180^\circ$, the caustic region below the end surface is crossed where the direct ray is shadowed by the cylinder.

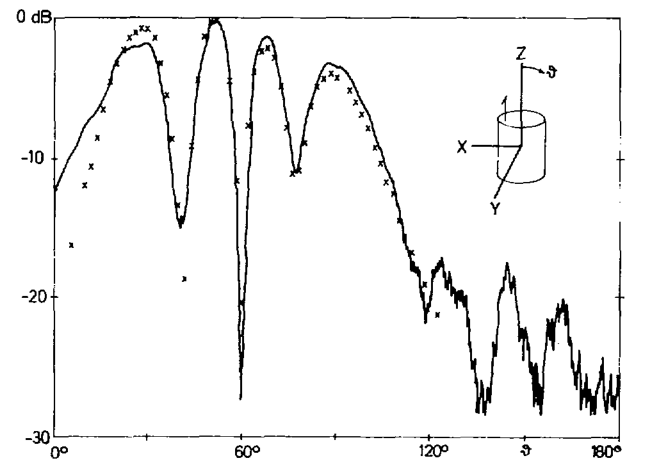


Fig. 6. Measured and computed fields across caustics away from cusp of caustics. — Measurement at 9.97 GHz on Cylinder model. xxxx Computation.

Unfortunately, the field is difficult to calculate in this region due to the fact that some of the singly diffracted rays are also shadowed by the cylinder. These rays are replaced by creeping waves on the cylinder excited at a point on the end surface. The position of this point may change significantly for small changes in the far-field direction and as a result, the caustic phenomena may interfere with penumbra-region phenomena. Neither this type of ray nor higher order diffracted rays were included in the computer program used, and no computation is shown in this region. Instead, Fig. 6 shows the measured pattern at $\phi = 166^\circ$ with $\theta$ varying from $0^\circ$ to $180^\circ$ to illustrate the field behavior in the lower region of space. The computed pattern up to $\theta = 120^\circ$ is also shown in Fig. 6 and a good agreement with the measurement is observed.

## VII. CONCLUSIONS

We have considered the field diffracted in a circular disk. When the source is placed away from the axis of the disk, the diffracted field has caustics which form diamond-shaped figures on the far-field sphere. Two sets of corrections to the GTD field have been derived. One set of corrections extends GTD to be valid across a caustic and the other set extends GTD to be valid across a cusp of caustics. The corrections are not restricted to the circular disk. In the general case, a correction will depend on the local properties of the diffracting edge at the point of diffraction.

The corrections are not valid when both the field point and the source point are close to the axis of the circular disk. In such cases, a substantial part of the edge may take part in the diffraction process, and the integral representation (1) of the diffracted field should be evaluated either numerically or analytically. Based on a number of computer runs, it was concluded that the caustic corrections derived in this paper should be utilized only when either the field point or the source point lies outside the two cones about the axis of the disk with cone angles equal to $\sqrt{2/ka}$ rad and apex at the center of the disk.

## APPENDIX A

## TRANSFORMATIONS OF PHASE INTEGRALS

In the case of two nearly coalescent diffraction points, $\phi' = \phi_2'$ and $\phi' = \phi_3'$, the phase integral

$$I = e^{-ikh(\phi_2')} \int_0^{2\pi} e^{ikh(\phi')} \, d\phi' \tag{A.1}$$

assumes the approximate form

$$I \sim \int_{-\infty}^{\infty} e^{ik(\frac{1}{2}h''(\phi_2')\phi'^2 + \frac{1}{6}h'''(\phi_2')\phi'^3)} \, d\phi' \tag{A.2}$$

where the integration limits have been extended to infinity, in recognition of the fact that the dominant contribution to the integral comes from the region close to the stationary point of the phase, and, furthermore, $\phi'$ has been replaced by $\phi' + \phi_2'$. When the substitution

$$\phi' = t - \frac{h''(\phi_2')}{h'''(\phi_2')} \tag{A.3}$$

is introduced into (A.2), the integral is transformed into

$$I \sim e^{\frac{ikh''(\phi_2')^3}{3h'''(\phi_2')^2}} \int_{-\infty}^{\infty} e^{ik(\frac{1}{6}h'''(\phi_2')t^3 - \frac{h''(\phi_2')^2}{2h'''(\phi_2')}t)} \, dt$$

$$= 2e^{\frac{ikh''(\phi_2')^3}{3h'''(\phi_2')^2}} \int_0^{\infty} \cos\left(k\left(\frac{1}{6}h'''(\phi_2')\, t^3 - \frac{h''(\phi_2')^2}{2h'''(\phi_2')}\, t\right)\right) dt$$

$$= 2\pi e^{i\frac{2}{3}\sigma^{\frac{3}{2}}} \sqrt[3]{\frac{2}{kh'''(\phi_2')}} \, \mathrm{Ai}\,(-\sigma) \tag{A.4}$$

where $\sigma$ is defined in (8), and Ai $(z)$ is the Airy function of first kind defined through

$$\mathrm{Ai}\,(z) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{1}{3}t^3 + zt\right) dt. \tag{A.5}$$

In the case of three equispaced, nearly coalescent diffraction points, the phase integral (A.1) assumes the approximate form

$$I \sim 2 \int_0^{\infty} e^{ik(\frac{1}{2}h''(\phi_2')\phi'^2 + \frac{1}{24}h^{(4)}(\phi_2')\phi'^4)} \, d\phi' \tag{A.6}$$

where assumptions similar to those leading to (A.2) have been made. Furthermore, we have utilized the fact that the integrand in (A.6) is an even function in $\phi'$. The substitution

$$\phi' = \sqrt{s} \sqrt[4]{\frac{12i}{kh^{(4)}(\phi_2')}} \tag{A.7}$$

is introduced into (A.6) which then becomes

$$I \sim 2 \sqrt[4]{\frac{3i}{4kh^{(4)}(\phi_2')}} \cdot \int_0^{\infty e^{-i\frac{\pi}{8}}} e^{ih''(\phi_2')\sqrt{\frac{3ik}{h^{(4)}(\phi_2')}}\, s - \frac{1}{2}s^2} s^{-\frac{1}{2}} \, ds. \tag{A.8}$$

Since the path of integration may be deformed to become the real axis without changing the value of the integral it follows that

$$I \sim \sqrt[4]{\frac{12}{kh^{(4)}(\phi_2')}} \sqrt{\pi}\, e^{i\frac{\pi}{8} - i\frac{3}{4}k\frac{h''(\phi_2')^2}{h^{(4)}(\phi_2')}} \cdot D_{-\frac{1}{2}}\left(h''(\phi_2') \sqrt{\frac{3k}{h^{(4)}(\phi_2')}}\, e^{-i\frac{\pi}{4}}\right) \tag{A.9}$$

where the parabolic cylinder function is defined through

$$D_{-\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{4}z^2} \int_0^{\infty} e^{-zs - \frac{1}{2}s^2} s^{-\frac{1}{2}} \, ds. \tag{A.10}$$

## APPENDIX B

## NUMERICAL EVALUATION OF CAUSTIC CORRECTIONS

The formulas for the caustic corrections contain either an Airy function Ai $(\sigma)$ or a parabolic cylinder function $D_{-1/2}(te^{-i\pi/4})$. In this Appendix we list the expressions for Ai $(\sigma)$ and $D_{-1/2}(te^{-i\pi/4})$ utilized in the computer program described in Section VI. These expressions represent a compromise between the accuracy and the speed of the computation. The Airy function of the caustic corrections (10) and (12) is evaluated by means of

$$\mathrm{Ai}\,(\sigma) = 0.35502805 f(\sigma) - 0.25881940\, g(\sigma) \tag{B.1}$$

with $f(\sigma)$ and $g(\sigma)$ defined by the series

$$f(\sigma) = 1 - \frac{\sigma^3}{2\cdot 3} + \frac{\sigma^6}{2\cdot 3\cdot 5\cdot 6} - \frac{\sigma^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots \tag{B.2}$$

and

$$g(\sigma) = \sigma - \frac{\sigma^4}{3\cdot 4} + \frac{\sigma^7}{3\cdot 4\cdot 6\cdot 7} - \frac{\sigma^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots \tag{B.3}$$

[11, p. 446]. When the series are truncated after the ninth- and the tenth-order terms, the resulting maximum error is less than 2 percent for $|\sigma| < 2$. While the Airy correction factor is put equal to unity for $\sigma < -1.2$, the caustic shadow term may require that the Airy function be evaluated for $\sigma > 2$. Then the asymptotic expression (14) is utilized. The resulting maximum error is less than 3 percent.

In the computer program, the cusp correction factors (21) and (22) are put equal to unity when the magnitude of the argument of $D_{-1/2}(te^{-i\pi/4})$ exceeds 2. For smaller arguments

$$D_{-\frac{1}{2}}(te^{-i\pi/4}) = 1.2162802\, y_1(t) - 0.5813683\, y_2(t) \tag{B.4}$$

is utilized with $y_1(t)$ and $y_2(t)$ defined by the alternating series

$$y_1(t) = 1 - \frac{1}{4}\frac{t^4}{3\cdot 4} + \frac{1}{4^2}\frac{t^8}{3\cdot 4\cdot 7\cdot 8} - \cdots \tag{B.5}$$

and

$$y_2(t) = te^{-i\pi/4}\left(1 - \frac{1}{4}\frac{t^4}{4\cdot 5} + \frac{1}{4^2}\frac{t^8}{4\cdot 5\cdot 8\cdot 9} - \cdots\right) \tag{B.6}$$

[11, pp. 686-687]. The maximum error due to truncation after the eighth-order terms is less than 1 percent for the absolute value of $t$ less than 2.

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## REFERENCES

[1] J. B. Keller, "Geometrical theory of diffraction," *J. Opt. Soc. Amer.*, vol. 52, pp. 116–130, 1962.
[2] D. S. Ahluwalia, R. M. Lewis, and J. Boersma, "Uniform asymptotic theory of diffraction by a plane screen," *SIAM J. Appl. Math.*, vol. 16, pp. 783–807, 1968.
[3] R. G. Kouyoumjian and P. H. Pathak, "A uniform geometrical theory of diffraction for an edge in a perfectly conducting surface," *Proc. IEEE*, vol. 62, pp. 1448–1461, 1974.
[4] J. B. Keller, "Diffraction by an aperture," *J. Appl. Phys.*, vol. 28, pp. 426–444, 1957.
[5] C. E. Ryan, Jr., and L. Peters, Jr., "Evaluation of edge-diffracted fields including equivalent currents for the caustic regions," *IEEE Trans. Antennas Propagat.*, vol. AP-17, pp. 292–299, 1969.
[6] E. F. Knott and T. B. A. Senior, "Comparison of three high-frequency diffraction techniques," *Proc. IEEE*, vol. 62, pp. 1468–1474, 1974.
[7] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves.* Englewood Cliffs, NJ: Prentice-Hall, 1973, pp. 382–421.
[8] W. V. T. Rusch and O. Sorensen, The geometrical theory of diffraction for axially symmetric reflectors," *IEEE Trans. Antennas Propagat.*, vol. AP-23, pp. 414–419, 1975.
[9] L. M. Brekhovskikh, *Waves in Layered Media.* New York: Academic Press, 1960.
[10] T. Pearcey, "The structure of an electromagnetic field in the neighbourhood of a cusp of a caustic," *Phil. Mag.*, vol. 37, pp. 311–317, 1946.
[11] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions.* New York: Dover, 1965.